UDC 531.31

CLASSIFICATION OF LINEAR INTEGRALS OF A HOLONOMIC MECHANICAL SYSTEM WITH n DEGREES OF FREEDOM

PMM Vol. 36, №1, 1972, pp. 125-129
I. ILIEV
(Plovdiv)
(Received March 17, 1971)

We have obtained certain theorems on the linear integrals of a mechanical system with n degrees of freedom and on their dependency on the solutions of the Killing equations and the coordinates of the generalized force.

We consider a mechanical system with n degrees of freedom with kinetic energy

$$T = {}^1/_2\, g_{\lambda\mu} q^{\lambda\cdot} q^{\mu\cdot}$$

and generalized forces $Q_\varkappa$. Here and subsequently a dot as a superscript denotes the derivative with respect to time t. The fulfillment of the conditions [1]

$$\nabla_s \xi_\varkappa + \nabla_\varkappa \xi_s = 0 \tag{1}$$

$$\xi^\varkappa Q_\varkappa = 0 \tag{2}$$

is necessary and sufficient for the existence of a linear integral of the system $\xi_\varkappa q^{\varkappa} = C$. The kinetic energy defines a space V_n with the metric

$$ds^2 = 2T\, dt^2 = g_{\lambda\mu} dq^\lambda dq^\mu$$

In this space every solution of the Killing equation (1) defines a one-parameter group of motions G_1. All solutions $\xi_a{}^\varkappa$ $(a = 1, 2, 3, \ldots, r)$ of Eq. (1) define a group G_r called the group of motions in V_n. After a transformation of group G_1 the point $M(q^i)$ goes into $M'(q'^i)$ whose coordinates are determined from the formula [2]

$$q'^i = q^i + \tau \xi^i(q) + \frac{\tau^2}{2!}\, \xi^j(q)\, \frac{\partial \xi^i(q)}{\partial q^j} + \ldots \tag{3}$$

Here τ is a parameter of the group. The collection of successive positions of the image $M(q^i)$ defines a trajectory of the group. An infinitesimal transformation of the group is given by the formula

$$q'^i = q^i + \xi^i(q)\, \delta\tau \tag{4}$$

Condition (2) shows: in order that $\xi_\varkappa q^\varkappa = c$ be a linear integral it is necessary that the displacement along a trajectory of group G_1 be perpendicular to the vector of generalized force with coordinates $Q^\varkappa$

Let us consider the case when all $Q_\varkappa = 0$. Then (2) is fulfilled identically and every solution of the Killing equation yields one linear integral of the system. This makes it possible to carry over to this case certain theorems from Eisenhart's monograph [2]. From Theorem 53.1 presented in [2] we obtain:

T h e o r e m 1. A mechanical system moving by inertia $(Q_\varkappa = 0)$ can have no more than ${}^1/_2\, n(n + 1)$ linear integrals and, moreover, the number of integrals equals ${}^1/_2\, n(n - 1)$, when V_n is a Riemann space with constant curvature.

From the Fubini theorem [2] we obtain:

Theorem 2. A mechanical system with $n > 2$ degrees of freedom, moving by inertia ($Q_\varkappa = 0$), cannot have $\frac{1}{2}n\,(n+1) - 1$ linear integrals.

Let us consider the case when at least one $Q_\varkappa \neq 0$. Let G_r be the group of motions in V_n and $\xi_a{}^\varkappa$, $a = 1,2,\ldots, r$, the system of vectors of the group. The group symbols are

$$X_a f = \xi_a{}^\varkappa\, \partial f/\partial q^\varkappa$$

Each vector $\xi^\varkappa = c^a \xi_a{}^\varkappa$, where c^a are constants, yields a one-parameter group of motions and also is a solution of the Killing equation (1). All solutions of this equation are yielded in just the same fashion. In this case condition (2) takes the form

$$c^a \xi_a{}^\varkappa Q_\varkappa = 0 \tag{5}$$

Every solution of Eq. (1), satisfying condition (5), yields a linear integral of the system.

Let us consider the case of a conservative mechanical system. Then relation (2) takes the form

$$\xi_a{}^\varkappa \frac{\partial u}{\partial q^\varkappa} = X_a u = 0 \tag{6}$$

As is known [2], for a suitable choice of the group parameter the trajectory equations have the form $dq^i / d\tau = \xi^i$ From condition (6) we obtain

$$\xi_a{}^\varkappa \frac{\partial u}{\partial q^\varkappa} = \frac{du}{d\tau} = 0 \tag{7}$$

This shows that if $\xi_a{}^\varkappa$ is a solution of Eq. (1), satisfying condition (2) in the form (7), then the trajectory of the group defined by $\xi_a{}^\varkappa$ is a line on the surface $u =$ const.

Let $\xi_a{}^\varkappa$, $m = 1,2,\ldots, p$ be a system of vectors satisfying the conditions indicated, which cannot be extended further. Let $\xi_a{}^\varkappa p_\varkappa = c_1$ and $\xi_b{}^\varkappa p_\varkappa = c_2$ be two linear integrals in which we have set $p_\varkappa = q_{\varkappa s} q^{\cdot s}$. For the group symbols X_a and X_b we have

$$[X_a, X_b]\, f = c^e_{ab} X_e f \quad (a,\ b,\ e = 1, 2, \ldots, r) \tag{8}$$

From (8) we obtain

$$\xi_a{}^\varkappa \frac{\partial \xi_b{}^s}{\partial q^\varkappa} - \xi_b{}^\varkappa \frac{\partial \xi_a{}^s}{\partial q^\varkappa} = c^e_{ab} \xi_e{}^s \tag{9}$$

where c^e_{ab} are the structure constants of G_r. As was shown in [2], $c^e_{ab}\xi_e{}^s$ is once again a vector of group G_r. Furthermore,

$$\left(\xi_a{}^\varkappa \frac{\partial \xi_b{}^s}{\partial q^\varkappa} - \xi_b{}^\varkappa \frac{\partial \xi_a{}^s}{\partial q^\varkappa}\right) u_s = \xi_a{}^\varkappa \frac{\partial \xi_b{}^s u_s}{\partial q^\varkappa} - \xi_b{}^\varkappa \frac{\partial \xi_a{}^s u_s}{\partial q^\varkappa} = 0 \tag{10}$$

$$u_s = \partial u/\partial q^s$$

Thus we have obtained that if $\xi_a{}^\varkappa p_\varkappa = c_1$ and $\xi_b{}^\varkappa p_\varkappa = c_2$ are linear integrals of the system, then $c^e_{ab}\xi^\varkappa_e p_\varkappa = c_3$ also is a linear integral.

For the function u we can write, from (8),

$$[X_a, X_b]\, u = c^e_{ab} X_e\ u \quad (a,\ b = 1, 2, \ldots, p;\ l = 1, 2, \ldots, r)$$

From (10) we have $[X_a, X_b]u = 0$. On the other hand, $X_m u = 0$ ($m = 1,2,\ldots,p$), so that $c^q_{ab} X_q u = 0$ ($q = p+1,\ldots, r$). The assumption that even one of the $c^q_{ab} \neq 0$ would lead to the conclusion that $c^q_{ab}\xi^\varkappa_q p_\varkappa = c$ is a linear integral and, consequently, $c^q_{ab}\xi^\varkappa_q$ can be represented as a linear combination of the vectors $\xi^\varkappa_m$, $m = 1, 2,\ldots,p$ with constant coefficients. Then

$$c_{ab}^{q}\xi_q^{\varkappa}=\theta_{ab}^{m}\xi_m^{\varkappa} \quad (m=1,2,\ldots,p;\ q=p+1,\ldots,r) \tag{11}$$

where θ_{ab}^{m} are constants. The latter is impossible since $\xi_a^{\varkappa}$ $(a=1,2,\ldots,r)$ are independent vectors with constant coefficients. Finally

$$[X_a, X_b]=c_{ab}^{f}X_f \quad (a,b=1,2,\ldots,r;\ f=1,2,\ldots,p) \tag{12}$$

Consequently, $\xi_1^{\varkappa}, \xi_2^{\varkappa},\ldots,\xi_p^{\varkappa}$ are the vectors of a subgroup G_p of group G_r. The order of group G_p determines the number of linear integrals of a conservative mechanical system. This group is intransitive and $u=\text{const}$ is its invariant manifold. The order p of the group does not exceed $^1/_2\ n(n-1)$. We say about the group G_p and about an arbitrary subgroup G_e of it that they are induced in G_r by the function $u=\text{const}$. Conversely, every intransitive subgroup of group G_r in V_n can be looked upon as an induced group under a suitable choice of the force function Φ

Let G_p be an intransitive subgroup of group G_r with the base vectors $\xi_1^{\varkappa},\xi_2^{\varkappa},\ldots,\xi_p^{\varkappa}$ By p_0 we denote the total rank of the matrix $M=\|\xi_f^{\varkappa}\|$ $(f=1,2,\ldots,p)$. Let $u_1(q)=c_1,\ u_2(q)=c_2,\ldots,\ u_{n-p_0}(q)=c_{n-p_0}$ be invariant manifolds of group G_p. Let us consider an arbitrary function $\Phi(u_1,u_2,\ldots,u_{n-p_0})$; $\xi_f^{\varkappa}$ are some of the vectors of group G_p. Then it is easy to see that condition (2) is fulfilled

$$\frac{\partial\Phi}{\partial u_\rho}\frac{\partial u_\rho}{\partial q^s}\xi_f^{s}=\frac{\partial\Phi}{\partial u_\rho}X_f u_\rho=0 \tag{13}$$

Since conditions (1) and (2) are fulfilled for an arbitrary vector $\xi_f^{\varkappa}$ from G_p, we see that the groups G_p indeed is induced in group G_r by the function $\Phi=\text{const}$.

Let us consider the motion of a point with mass $m=1$ in a three-dimensional Euclidean space. In this case the group of motions is six-parametric. Its vectors may be

$$\xi_1(1,0,0),\quad \xi_2(0,1,0),\quad \xi_3(0,0,1) \tag{14}$$

$$\xi_4(-y,x,0),\quad \xi_5(z,0,-x),\quad \xi_6(0,-z,y)$$

The vectors ξ_4, ξ_5, ξ_6 define an intransitive group called the rotation group. The total rank of the matrix formed by them [2] equals two and, consequently, there exists one invariant manifold, namely, $x^2+y^2+z^2=c$. Then, if the force function has the form $u(x^2+y^2+z^2)$, the mechanical system has the linear integrals

$$-yx^{\cdot}+xy^{\cdot}=c_1,\quad zx^{\cdot}-xz^{\cdot}=c_2\ -zy^{\cdot}+z^{\cdot}y=c_3$$

Everything we have said for the conservative system can be carried over to mechanical systems for which

$$Q_{\varkappa}=\rho\partial u/\partial q^{\varkappa} \tag{15}$$

Indeed, for such systems conditions (1) and (2), after cancelling ρ in Eq. (15), lead to corresponding conditions for the system in which $Q_{\varkappa}=\partial u/\partial q^{\varkappa}$. Consequently, both systems have like linear integrals. If $Q_{\varkappa}$ are the coordinates of the generalized force, then as is well known [3], for $Q_{\varkappa}$ to have form (15) it is necessary and sufficient to fulfill the condition

$$Q\,\text{rot}\,Q=0,\qquad Q=(Q_1,\ Q_2,\ Q_3) \tag{16}$$

Let us consider the case when G_e is an Abelian group. Then we can make a change of variables [2] so that

$$\xi_\rho^{\varkappa}=\delta_\rho^{\varkappa} \quad (\rho=1,2,\ldots,e;\ \chi=1,2,\ldots,n) \tag{17}$$

We write condition (1) in the form

$$\xi_\rho^{\varkappa} \frac{\partial g_{ij}}{\partial q^{\varkappa}} + g_{i\varkappa} \frac{\partial \xi_\rho^{\varkappa}}{\partial q^j} + g_{\varkappa j} \frac{\partial \xi_\rho^{\varkappa}}{\partial q^i} = 0 \tag{18}$$

Having substituted $\xi_\rho^{\varkappa}$ from (17), we find

$$\partial g_{ij} / \partial q^{\rho} = 0 \tag{19}$$

We obtained that $g_{ij} = g_{ij}(q^{e+1}, \ldots, q^n)$. From condition (2) we finally have

$$\partial u/\partial q^{\rho} = 0, \qquad \partial T/\partial q^{\rho} = 0 \qquad (\rho = 1, 2, \ldots, e) \tag{20}$$

Condition (20) shows that $q^1, q^2, \ldots, q^e$ are ignorable coordinates [4]. From the results of [1] it follows that if $q^1, q^2, \ldots, q^e$ are ignorable coordinates, then $\xi_p^{\varkappa} = \delta_p^{\varkappa}$. By Theorem 51.6 of [2] the latter define an Abelian group of motions since $[X_a, X_b] = 0$. Thus, we have proved the following theorem:

Theorem 3. For the possibility of a correspondence of the linear integrals of a given mechanical system with ignorable coordinates, it is necessary and sufficient that the group G_e induced by the function $u = \text{const}$, corresponding to these integrals, be Abelian group.

In the example analyzed above let $u = u(x^2 + y^2)$. Then ξ_3 and ξ_4 define a subgroup G_2 induced by $u = \text{const}$. Since $[X_3, X_4] = 0$, this group is Abelian group. The linear integrals have the form

$$\dot{z} = c_1 \qquad x\dot{y} - y\dot{x} = c_2 \tag{21}$$

In accordance with Theorem 3 we can choose new parameters for which $\xi^{\varkappa}{}_a = \delta^{\varkappa}{}_a$ (it suffices to make the change of variables $x = r\cos\varphi$ and $y = r\sin\varphi$).

Corollary. By a suitable change of variables every linear integral can be transformed into an ignorable coordinate.

This assertion is well-known in analytical mechanics as Lévy's theorem, proved in [4] with the aid of the theory of contact transformations.

Let us consider a mechanical system with three degrees of freedom. The expression for kinetic energy defines a Riemann space V_3. Let it admit of a group G_2. Then we can take the line element in the form [2]

$$ds^2 = 2T dt^2 = g_{ij} dq^i dq^j + (dq^3)^2 \qquad (i, j = 1, 2) \tag{22}$$

Let $u = u(q^3)$. Then group G_2 can be looked upon as having been induced by the function $u(q^3) = \text{const}$. The following two cases are possible:

1. G_2 is an Abelian group. As follows from Theorem 3, the group vectors may be $\xi_1(1, 0, 0)$, $\xi_2(0, 1, 0)$. In this case $g_{\lambda\mu} = g_{\lambda\mu}(q^3)$ and, since $u = u(q^3)$, we have the ignorable coordinates q^1 and q^2.

2. G_2 is not an Abelian group. In this case the relations

$$[X_1, X_2]f = X_1 f$$
$$q_{11} = \alpha, \quad g_{12} = \alpha q^1 + \beta, \quad g_{22} = \alpha (q^1)^2 + \beta q^1 + \gamma \tag{23}$$

are fulfilled, where α, β, γ depend on q^3. In this case the vectors of the group are $\xi_1(e^{-q^2}, 0, 0)$, $\xi_2(0, 1, 0)$. The system has the two linear integrals

$$p_1 \exp(-q^2) = c, \qquad p_2 = c_2 \tag{24}$$

We see that the second coordinate is ignorable while the first is a latent ignorable coordinate. Note that in Case 2 it is impossible to make a change of variables such that both linear integrals would be ignorable coordinates. As is well known, for this it is necessary that the group be transformable to an Abelian group. On the other hand, every group can be transformed by a change of variables to a group similar to it [2], and, by a suitable choice of base vectors, the similar groups will have like structure constants. According to the above-cited corollary there exists a change of coordinate systems transforming the first coordinate to an ignorable coordinate, but here the second coordinate becomes a latent ignorable coordinate.

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Translated by NHC

UDC 534.013

RESONANCE OSCILLATIONS OF A COMPOUND TORSION PENDULUM

PMM Vol. 36, №1, 1972, pp. 129-138

B. I. CHESHANKOV
(Sofia)

(Received December 10, 1970)

The oscillations of conservative systems with two degrees of freedom under internal resonance were examined in [1 - 6]. We investigate the resonance oscillations of one mechanical system and ascertain the features of its behavior.

1. Consider the system shown in Fig. 1. It consists of a disk attached to a thin elastic spindle having a coefficient of elasticity c. A compound pendulum rotates around an axis $O\eta$ belonging to the disk and perpendicular to the disk's axis of rotation (in the Figure this axis is perpendicular to the plane of the diagram). We take it that ξ, η, ζ are the principal inertial axes and that the compound pendulum has the principal moments of inertia I_ξ, I_η, I_ζ with respect to them. I is the disk's moment of inertia with respect to the axis of rotation. We denote the pendulum's center of gravity by C ; the distance $OC = e$, φ_1 is the disk's angle of rotation from the equilibrium position, φ_2 is the pendulum's angle of deviation from the vertical, m is the mass of the pendulum. In this notation we have:

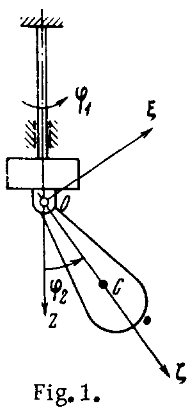

Fig. 1.

for the system's kinetic energy,

$$T = {}^1/_2 (I + I_\xi \sin^2 \varphi_2 + I_\varphi \cos^2 \varphi_2) \varphi_1^{\cdot 2} + {}^1/_2 I_\eta \varphi_2^{\cdot 2} \qquad (1.1)$$

for the system's potential energy,